Problem A:

Derive the equation of motion of the system shown in the figure below.

What is the damping ratio and the natural frequency of the system?

Problem B:

Consider the spring mass system shown below. The spring has a mass density of \( \delta \) kg/m in its un-deformed state, and an un-deformed length of \( L \). Derive the equations of motion and determine the natural frequency of the system.
Problem C:

Consider the following partitioned system matrix:

\[ A = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} \]

where \( A_{11}(t), A_{22}(t) \) and \( A_{12}(t) \) are generally time-varying sub-matrices (they cannot be assumed to be scalars).

(a) Prove that the state transition matrix has the form

\[ \Phi(t, t_0) = \begin{bmatrix} \Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\ 0 & \Phi_{22}(t, t_0) \end{bmatrix} \]

where \( \frac{\partial}{\partial t} \Phi_{ii}(t, t_0) = A_{ii}(t) \Phi_{ii}(t, t_0), i = 1, 2 \) and

\( \frac{\partial}{\partial t} \Phi_{12}(t, t_0) = A_{11}(t) \Phi_{12}(t, t_0) + A_{12}(t) \Phi_{22}(t, t_0) \), with \( \Phi_{12}(t_0, t_0) = 0 \).

(b) Use this result to find \( \Phi(t, 0) \) for

\[ A = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \]

Prove that your answer is correct.
Problem D:

Consider the four-link mechanism shown in the figure below. Determine the maximum displacement of link 4.

The mechanism is driven by a D.C. motor attached to the crank with an angular velocity of 60 rpm clockwise. Determine the velocity of the link 4 when the crank rotates at 60 rpm for the link locations shown below.

Problem E:

Consider the following open-loop system:

\[
\frac{C(s)}{U(s)} = \frac{1}{s^2 - 1}
\]

- With an impulse input for \( u(t) \) and zero initial conditions, solve for \( c(t) \).
- Draw the Bode plot for the open-loop system.
- Suppose we multiply the open-loop transfer function by a constant. What does this do to the magnitude and phase of the Bode plot?
- Can the system be controlled from an asymptotic sense using proportional control only? Explain this answer using 1) a phase margin argument, as well as 2) from the closed-loop transfer.
- Design a compensator \( G_C \) to stabilize the system so that the closed-loop system has a natural frequency of 2 rad/sec and a damping ratio of \( \sqrt{2} / 2 \).
Problem F:

A linear, time-invariant, SISO (single input – single output) system is represented by the Bode plots shown below. Answer the following questions:

a) Estimate the system's transfer function
b) Is this system stable, marginally stable, or unstable? Explain.

c) What is the system's output if the input is \( u = 5 \times \sin(4t) \)?

d) If the input is \( u = 10 \times \sin(w^2t) \), for what value(s) of \( w \) (if any) is the system output magnitude exactly equal to 10?

e) If the input is \( u = 10 \times \sin(w^2t) \), for what value(s) of \( w \) (if any) is the system output magnitude exactly equal to 1?

f) If the input is \( u = 10 \times \sin(w^2t) \), for what value(s) of \( w \) (if any) is the system output magnitude exactly equal to 100?

g) If the input is \( u = 10 \times \sin(w^2t) \), what is the greatest possible output magnitude?

h) If the input is \( u = 10 \times \sin(w^2t) \), for what value of \( w \) does the greatest output magnitude occur?

i) If the input is \( u = 10 \times \sin(w^2t) \), for what value(s) of \( w \) (if any) does the system act as an amplifier?

j) What is the system's approximate natural frequency (if any)?
What is the system's approximate damping ratio?

Bode Diagrams

[Graph showing Bode plots for phase and magnitude vs. frequency.]
EXAMINATION

MECHANICAL AND AEROSPACE ENGINEERING

2008 PhD. QUALIFYING EXAM
NO TEXTBOOKS OR NOTES ALLOWED. YOU ARE REQUIRED TO ANSWER 4 OF THE 6 QUESTIONS.

**Problem A:**

Derive the equation of motion of the system shown in the figure below.

What is the damping ratio and the natural frequency of the system?

**Problem B:**

Consider the spring mass system shown below. The spring has a mass density of $8$ kg/m in its un-deformed state, and an un-deformed length of $L$. Derive the equations of motion and determine the natural frequency of the system.
Problem C:

Consider the following partitioned system matrix:

\[ A = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} \]

where \( A_{11}(t) \), \( A_{22}(t) \) and \( A_{12}(t) \) are generally time-varying sub-matrices (they cannot be assumed to be scalars).

a) Prove that the state transition matrix has the form

\[ \Phi(t,t_0) = \begin{bmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ 0 & \Phi_{22}(t,t_0) \end{bmatrix} \]

where \( \frac{\partial}{\partial t} \Phi_{ii}(t,t_0) = A_{ii}(t) \Phi_{ii}(t,t_0), i=1,2 \) and

\( \frac{\partial}{\partial t} \Phi_{12}(t,t_0) = A_{11}(t) \Phi_{12}(t,t_0) + A_{12}(t) \Phi_{22}(t,t_0) \), with \( \Phi_{12}(t_0,t_0) = 0 \).

b) Use this result to find \( \Phi(t,0) \) for

\[ A = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \]

Prove that your answer is correct.
**Problem D:**

Consider the four-link mechanism shown in the figure below.
Determine the maximum displacement of link 4.

The mechanism is driven by a D.C. motor attached to the crank with an angular velocity of 60 rpm clockwise.
Determine the velocity of the link 4 when the crank rotates at 60 rpm for the link locations shown below.

![Diagram of a four-link mechanism](image)

**Problem E:**

Consider the following open-loop system:

\[
\frac{C(s)}{U(s)} = \frac{1}{s^2 - 1}
\]

- With an impulse input for \( u(t) \) and zero initial conditions, solve for \( c(t) \).
- Draw the Bode plot for the open-loop system.
- Suppose we multiply the open-loop transfer function by a constant. What does this do to the magnitude and phase of the Bode plot?
- Can the system be controlled from an asymptotic sense using proportional control only? Explain this answer using 1) a phase margin argument, as well as 2) from the closed-loop transfer.
- Design a compensator \( G_c \) to stabilize the system so that the closed-loop system has a natural frequency of 2 rad/sec and a damping ratio of \( \sqrt{2}/2 \).
Problem F:

A linear, time-invariant, SISO (single input – single output) system is represented by the Bode plots shown below. Answer the following questions:

a) Estimate the system’s transfer function
b) Is this system stable, marginally stable, or unstable? Explain.
c) What is the system’s output if the input is \( u = 5 \sin(4t) \)?
d) If the input is \( u = 10 \sin(w^t) \), for what value(s) of \( w \) (if any) is the system output magnitude exactly equal to 10?
e) If the input is \( u = 10 \sin(w^t) \), for what value(s) of \( w \) (if any) is the system output magnitude exactly equal to 1?
f) If the input is \( u = 10 \sin(w^t) \), for what value(s) of \( w \) (if any) is the system output magnitude exactly equal to 100?
g) If the input is \( u = 10 \sin(w^t) \), what is the greatest possible output magnitude?
h) If the input is \( u = 10 \sin(w^t) \), for what value of \( w \) does the greatest output magnitude occur?
i) If the input is \( u = 10 \sin(w^t) \), for what value(s) of \( w \) (if any) does the system act as an amplifier?
j) What is the system’s approximate natural frequency (if any)?
What is the system's approximate damping ratio?

Bode Diagrams
EXAMINATION

MECHANICAL AND AEROSPACE ENGINEERING

2008 PhD. QUALIFYING EXAM
Problem A:

Derive the equation of motion of the system shown in the figure below.

What is the damping ratio and the natural frequency of the system?

Problem B:

Consider the spring mass system shown below. The spring has a mass density of $\delta$ kg/m in its un-deformed state, and an un-deformed length of $L$. Derive the equations of motion and determine the natural frequency of the system.
Problem C:

Consider the following partitioned system matrix:

\[ A = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} \]

where \( A_{11}(t) \), \( A_{22}(t) \) and \( A_{12}(t) \) are generally time-varying sub-matrices (they cannot be assumed to be scalars).

a) Prove that the state transition matrix has the form

\[ \Phi(t, t_0) = \begin{bmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ 0 & \Phi_{22}(t,t_0) \end{bmatrix} \]

where \( \frac{\partial}{\partial t} \Phi_{ii}(t,t_0) = A_{ii}(t)\Phi_{ii}(t,t_0), i = 1,2 \) and

\( \frac{\partial}{\partial t} \Phi_{12}(t,t_0) = A_{11}(t)\Phi_{12}(t,t_0) + A_{12}(t)\Phi_{22}(t,t_0) \), with \( \Phi_{12}(t_0,t_0) = 0 \).

b) Use this result to find \( \Phi(t,0) \) for

\[ A = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \]

Prove that your answer is correct.
Problem D:

Consider the four-link mechanism shown in the figure below. Determine the maximum displacement of link 4.

The mechanism is driven by a D.C. motor attached to the crank with an angular velocity of 60 rpm clockwise. Determine the velocity of the link 4 when the crank rotates at 60 rpm for the link locations shown below.

Problem E:

Consider the following open-loop system:

\[
\frac{C(s)}{U(s)} = \frac{1}{s^2 - 1}
\]

- With an impulse input for \( u(t) \) and zero initial conditions, solve for \( c(t) \).
- Draw the Bode plot for the open-loop system.
- Suppose we multiply the open-loop transfer function by a constant. What does this do to the magnitude and phase of the Bode plot?
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Problem F:

A linear, time-invariant, SISO (single input - single output) system is represented by the Bode plots shown below. Answer the following questions:

a) Estimate the system’s transfer function
b) Is this system stable, marginally stable, or unstable? Explain.
c) What is the system’s output if the input is $u = 5 \times \sin (4t)$?
d) If the input is $u = 10 \times \sin (w t)$, for what value(s) of $w$ (if any) is the system output magnitude exactly equal to 10?
e) If the input is $u = 10 \times \sin (w t)$, for what value(s) of $w$ (if any) is the system output magnitude exactly equal to 1?
f) If the input is $u = 10 \times \sin (w t)$, for what value(s) of $w$ (if any) is the system output magnitude exactly equal to 100?
g) If the input is $u = 10 \times \sin (w t)$, what is the greatest possible output magnitude?
h) If the input is $u = 10 \times \sin (w t)$, for what value of $w$ does the greatest output magnitude occur?
i) If the input is $u = 10 \times \sin (w t)$, for what value(s) of $w$ (if any) does the system act as an amplifier?
j) What is the system’s approximate natural frequency (if any)?
What is the system's approximate damping ratio?

Bode Diagrams

[Graph showing Bode plot with axes labeled Phase (deg), Magnitude (dB) and Frequency (rad/sec).]
Problem A:

Derive the equation of motion of the system shown in the figure below.

What is the damping ratio and the natural frequency of the system?

Problem B:

Consider the spring mass system shown below. The spring has a mass density of δ kg/m in its un-deformed state, and an un-deformed length of L. Derive the equations of motion and determine the natural frequency of the system.
**Problem C:**

Consider the following partitioned system matrix:

\[ A = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} \]

where \( A_{11}(t) \), \( A_{22}(t) \) and \( A_{12}(t) \) are generally time-varying sub-matrices (they cannot be assumed to be scalars).

a) Prove that the state transition matrix has the form

\[ \Phi(t,t_0) = \begin{bmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ 0 & \Phi_{22}(t,t_0) \end{bmatrix} \]

where \( \frac{\partial}{\partial t} \Phi_{ii}(t,t_0) = A_{ii}(t) \Phi_{ii}(t,t_0) \), \( i = 1,2 \) and
\( \frac{\partial}{\partial t} \Phi_{12}(t,t_0) = A_{11}(t) \Phi_{12}(t,t_0) + A_{12}(t) \Phi_{22}(t,t_0) \), with \( \Phi_{12}(t_0,t_0) = 0 \).

b) Use this result to find \( \Phi(t,0) \) for

\[ A = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \]

Prove that your answer is correct.
Problem D:

Consider the four-link mechanism shown in the figure below. Determine the maximum displacement of link 4.

The mechanism is driven by a D.C. motor attached to the crank with an angular velocity of 60 rpm clockwise. Determine the velocity of the link 4 when the crank rotates at 60 rpm for the link locations shown below.

![Mechanism Diagram]

Problem E:

Consider the following open-loop system:

\[ \frac{C(s)}{U(s)} = \frac{1}{s^2 - 1} \]

- With an impulse input for \( u(t) \) and zero initial conditions, solve for \( c(t) \).
- Draw the Bode plot for the open-loop system.
- Suppose we multiply the open-loop transfer function by a constant. What does this do to the magnitude and phase of the Bode plot?
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a) Estimate the system's transfer function
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g) If the input is $u = 10 \sin(w^t)$, what is the greatest possible output magnitude?
h) If the input is $u = 10 \sin(w^t)$, for what value of $w$ does the greatest output magnitude occur?
i) If the input is $u = 10 \sin(w^t)$, for what value(s) of $w$ (if any) does the system act as an amplifier?
j) What is the system's approximate natural frequency (if any)?
What is the system’s approximate damping ratio?

Bode Diagrams

Phase (deg) vs. Magnitude (dB)

Frequency (rad/sec)

10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
-110
-120
-130
-140
-150
-160
-170
-180
10^0
10^1