Closed Book

**Computation and Applied Mechanics Qualifier, Sp 2009**

**Please work all 6 problems**; time allowed is 3 hrs, so divide your time well among the problems—do not spend too much time on any one problem!

1) Consider a beam theory where plane sections in the yz plane remain plane after bending and bending is restricted to one plane. The displacements are assumed to be given by $w(y,z) = \psi(z)y, \quad v = v(z)$ where ψ is the rotation angle of the cross-section.


How many unknown displacement functions are there at this point? ______

Does this theory have any through-thickness stretch? Demonstrate.

For Euler-Bernoulli (E-B) beam theory, there is an additional assumption about $\gamma_{yz}$. Show how this assumption leads to a reduction in the number of displacement functions.

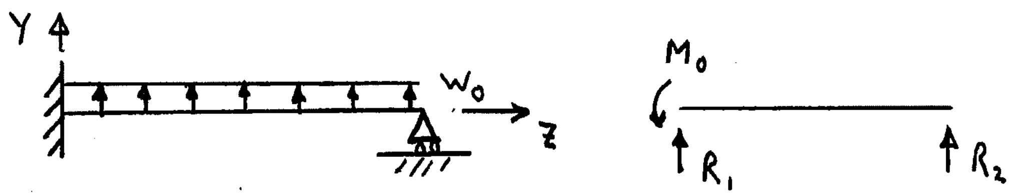
Use the definition $M = \iint_A \sigma_z y dA$ and a one-dimensional constitutive equation $\sigma_z = E\varepsilon_z$ to derive a moment-curvature relationship for the E-B beam.

Derive the moment and shear equilibrium equations for the beam by considering an infinitesimal length "dz"; see the figure below for sign definitions.


If the beam has an axial load "P" and a slope $\frac{dy}{dz}$, show how this will affect the moment equilibrium equation by contributing an extra term.

In the Timoshenko beam theory, we permit a non-zero value of $\gamma_{yz}$. How many displacement variables will there be using the same displacement assumptions as given above?

2) Consider the beam of length "L" shown below. Is this beam statically determinate? Demonstrate.




a) In this system, the ODE for an Euler-Bernoulli beam is $-EIv'' = M(z)$ .

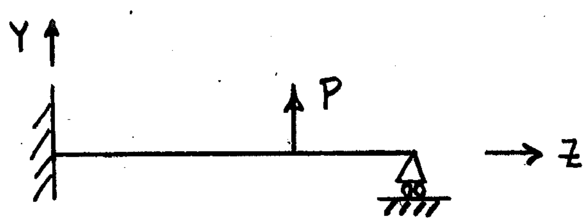
Simply assume that you know the reaction force $R_2$ (beam is assumed to be a cantilever), formulate M(z), and integrate it to get the deflection . Show which b.c. on displacement and its derivatives you would use to solve for the unknown coefficients. Do not actually solve for those coefficients.

You still have an unknown value, mainly the reaction $R_2$ . How would you solve for this value? Do not actually solve.

b) Instead of using an exact solution, we propose an approximation solution to the above beam loading, given by $v(z) = v_0 z^2 (L - z)$ . Demonstrate that this approximation satisfies all the essential b.c.

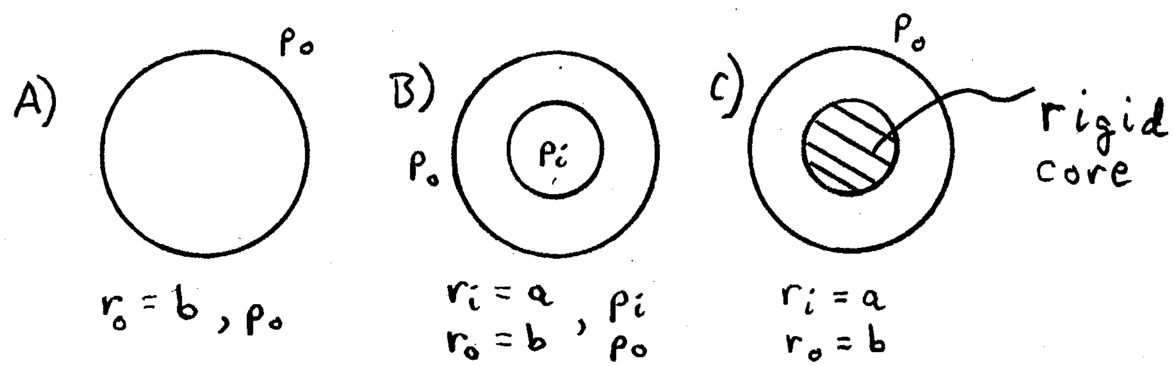
Using the Theorem of Minimum Potential Energy, completely set up the expression for Π and indicate how you would solve for the undetermined coefficient $v_0$ .Will this be a linear or higher order equation for $v_0$ ? Do not actually solve.

What would be the work term if the loading were a single point loading at z=a as shown:




3) Consider an axisymmetric cylinder.

a) Write the conditions you would pose and show how these could be used to solve for the coefficients $C_1$ and $C_2$ for u(r) for each of the three conditions below, which have inner and outer radii and pressures as given. **You do not actually need to complete the solution for the coefficients.**




b) If there is a single cylinder of format "B" with inner pressure equal to zero and there are no end caps, consider the shape of the curves for the stresses $\sigma_r and\ \sigma_\theta$ .

Use the Tresca criterion to predict **where you would check for yield and which stresses you would consider**. (Do not solve for a critical value of p0.) Hint: Assume say (a/b)=1/2 and make a small sketch which shows values at r=a and r=b and connect them for each stress.

c) Let us add end caps and still use the Tresca criterion to predict failure. Does adding the end caps make any difference to which stresses we would consider r for the critical value of pressure or to the critical value itself? Demonstrate.

d) The term $\frac{E}{1-\nu^2}$ indicates an assumption of plane stress. If the cylinder is long, plane strain would be more appropriate. Please give the assumptions of plane strain and demonstrate how you would produce a constitutive equation which relates normal stresses & strains in the r-θ plane.

**Problem 4**

Consider a linear elastic isotropic material in three-dimensions with the constitutive relationship given as

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk} ,$$

where $\sigma_{ij}$ and $\epsilon_{ij}$ represent components of the stress and strain tensors, respectively, while $\mu$ and $\lambda$ are the Lame elastic moduli. Assume standard indicial notation with sums over repeated indices.

(a) Find the relationship between the principal stresses and the principal strains.
(b) Show that the principal directions of stress and strain always coincide.

**Problem 5**

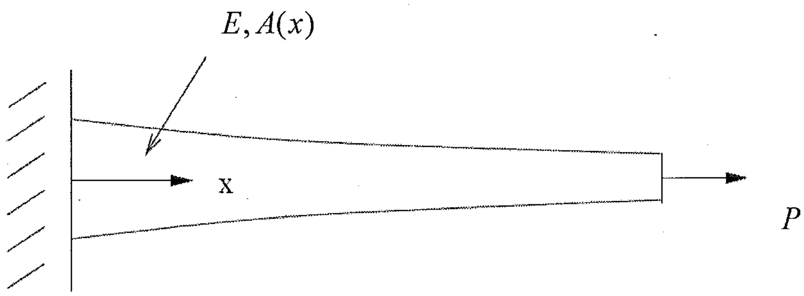
A tapered elastic bar has length $L$ and cross-sectional area $A(x)$, as shown in the figure below. Assume that the response is governed by the following one-dimensional differential equation of equilibrium:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+f=0,$$

where $E$ is the elastic modulus, $u(x)$ is the axial displacement and $f(x)$ is an applied body force per unit length. The bar is restrained at $x=0$, while normal tractions are applied at $x=L$ as illustrated, such that

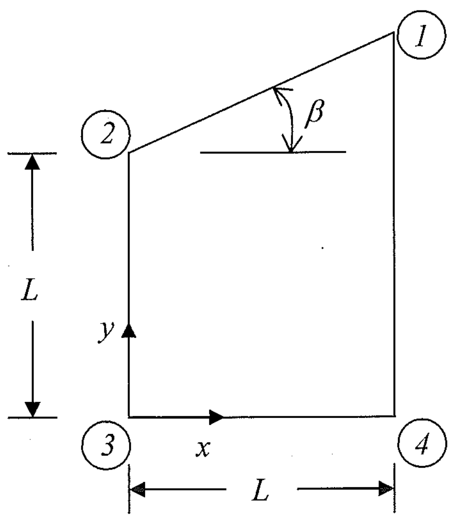
$$EA\frac{du}{dx}\bigg|_{x=L}=P.$$

Derive a weak form for this problem in two different ways, using (a) the principle of virtual work and (b) the principle of minimum total potential energy for the bar.



**Problem 6**

Determine the Jacobian matrix for the two-dimensional four-node element shown below. Demonstrate that this Jacobian is constant for the case with $\beta = 0^{o}$, but is singular for $\beta = -45^{o}$.



## SOME TENSOR OPERATIONS

### Transformation

$$x_{p'} = a_{p'i}\, x_i$$

$$x_i = a_{ip'}\, x_{p'}$$

### Orthogonality

$$a_{p'i}\, a_{q'i} = \delta_{p'q'} \;\; , \; a_{ip'}\, a_{jp'} = \delta_{ij}$$

### Alternating tensor

$$e_{ijk} = \frac{1}{2}\,(i - j\,)\,(j - k)\,(k - l)$$

$$e_{ijk}\, e_{kqr} = \delta_{iq}\, \delta_{jr} - \delta_{ir}\, \delta_{jq}$$

Notation for unit vectors: $\hat{e}_i$ , $\hat{e}_{p'}$

### Cross-product

$$\vec{C} = \vec{A}\, x\, \vec{B} \;\; is \;\; C_i = e_{ijk}\, A_j\, B_k$$

### Gradient

$$\nabla\,(\cdot) = \frac{\partial}{\partial x_i}\,(\cdot)$$

# MAE 415 F 2008 Test 1 equations

*Stress Transformation Equations*

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{x'y'} = \tau_{xy}(\cos^2\theta - \sin^2\theta) + (\sigma_y - \sigma_x)\sin\theta\cos\theta$$

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

$$[\sigma'] = [T][\sigma][T]^T$$

*Determining Principal Stresses*

$$\sigma_{\max,\min} = \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

*Strain Definitions*

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

*Strain Transformation Equations*

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\gamma_{x'y'} = -(\varepsilon_x - \varepsilon_y)\sin 2\theta + \gamma_{xy}\cos 2\theta$$

*Determining Principal Strains*

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

*von Mises Criterion*

$$(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6({\tau^2}_{xy} + {\tau^2}_{yz} + {\tau^2}_{xz}) = 2{\sigma^2}_{yp}$$

*Generalized Hooke's Law*

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

*Thermal Strain in Free Expansion*

$$\varepsilon_t = \alpha \Delta T$$

*Area of a Circle*

$$A = \pi r^2$$

Equation sheet, MAE 415, test 2 F08

*Compatibility Equation for Stress Functions*

$$\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=\nabla^4\Phi=0$$

*Relating Stress Functions to Stresses*

$$\sigma_x=\frac{\partial^2\Phi}{\partial y^2}\qquad \sigma_y=\frac{\partial^2\Phi}{\partial x^2}$$

$$\tau_{xy}=-\frac{\partial^2\Phi}{\partial x\partial y}$$

Or, in (r,θ),

$$\sigma_r=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$$

$$\sigma_\theta=\frac{\partial^2\Phi}{\partial r^2}$$

$$\tau_{r\theta}=\frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\Phi}{\partial r\,\partial\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)$$

*Generalized Hooke's Law-isothermal*

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}=\begin{bmatrix}\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{Bmatrix}$$

von Mises yield criterion, plane stress, in principal coordinates

$$\sigma_{eq}=sqrt\,(\sigma_1{}^2+\sigma_2{}^2-\sigma_1\sigma_2)=\sigma_{yp}$$

*General solution, axisymmetric, isothermal annulus*

$$u=c_1r+\frac{c_2}{r}\ ;\ \varepsilon_r=\frac{u}{r}\ ,\varepsilon_\theta=u'(r)$$

$$\sigma_r=\frac{E}{1-\nu^2}\left[c_1\,(1+\nu)\ -\ c_2\,\frac{(1-\nu)}{r^2}\right]$$

*Annulus, internal & external pressure*

$$\sigma_r=\frac{a^2p_i-b^2p_o}{b^2-a^2}-\frac{(p_i-p_o)\,a^2b^2}{(b^2-a^2)\,r^2}$$

$$\sigma_\theta=\frac{a^2p_i-b^2p_o}{b^2-a^2}+\frac{(p_i-p_o)\,a^2b^2}{(b^2-a^2)\,r^2}$$

$$u=\frac{1-\nu}{E}\frac{(a^2p_i-b^2p_o)\,r}{b^2-a^2}+\frac{1+\nu}{E}\frac{(p_i-p_o)\,a^2b^2}{(b^2-a^2)\,r}$$

If closed-ended cylinder under pressure, add

$$\sigma_z=\frac{p_ia^2-p_ob^2}{b^2-a^2}$$

*Compound cylinder, cylinder 2 over cylinder 1*

$$p=\frac{\delta}{b\left[\frac{1}{E_2}\left(\frac{c^2+b^2}{c^2-b^2}+\nu_2\right)+\frac{1}{E_1}\left(\frac{b^2+a^2}{b^2-a^2}-\nu_1\right)\right]}$$

*Rotating annulus*

$$\sigma_r=\frac{3+\nu}{8}\left(a^2+b^2-r^2-\frac{a^2b^2}{r^2}\right)\rho\varpi^2$$

$$u=\frac{(3+\nu)\,(1-\nu)}{8E}\left(a^2+b^2-\frac{1+\nu}{3+\nu}r^2+\frac{1+\nu}{1-\nu}\frac{a^2b^2}{r^2}\right)\rho\varpi^2r$$

*Thermal stress in annulus*

General solution:

$$u=\frac{(1+\nu)\alpha}{(1-\nu)r}\int_a^r \mathrm{T}r\,dr+c_1r+\frac{c_2}{r}$$

$$\sigma_r=\frac{\mathrm{E}}{(1+\nu)}\left[-\frac{(1+\nu)\alpha}{(1-\nu)r^2}\int_{\mathrm{a}}^{\mathrm{r}}\mathrm{Tr\,dr}+\frac{c_1}{(1-2\nu)}-\frac{c_2}{r^2}\right]$$

# MAE 415 Test #3 Equation Sheet, Fall 2008

Beam Centroid

$$\bar{z} = \frac{\Sigma A_i \bar{z}_i}{\Sigma A_i}$$

Product of inertia

$$I_{zz} = \Sigma \left(I_{zz} + A{d_y}^2\right)_i$$
$$I_{yz} = \Sigma \left(I_{yz} + A d_y d_z\right)_i$$

Moment-curvature

$$\frac{d^2 v}{dx^2} = \frac{M_z}{EI_z}$$

Bending stress

$$\sigma_x = \frac{(M_y I_z + M_z I_{yz})z - (M_y I_{yz} + M_z I_y)y}{I_y I_z - I_{yz}^2}$$

Displacement assumption

$$u = -y\frac{dv}{dx} + u_0(x)$$

Beam statics equations

$$\frac{dV}{dx} = -p$$

$$\frac{dM}{dx} = -V$$

Composite beam

$$\bar{y} = \frac{A_1 \bar{y}_1 + \Sigma n_i A_i \bar{y}_i}{A_1 + \Sigma n_i A_i}, \qquad n_i = \frac{E_i}{E_1}$$

$$I_t = I_1 + \Sigma n_i I_i$$

$$\sigma_{x1} = -\frac{My}{I_t}, \qquad \sigma_{xi} = -\frac{n_i My}{I_t}$$

Reciprocity Theorem

$$\sum_{k=1}^{m} P_k' \delta_k'' = \sum_{j=1}^{n} P_j'' \delta_j'$$

Typical strain energy contributions in beam

$$U = \int \frac{N^2 dx}{2AE} + \int \frac{M^2 dx}{2EI} + \int \frac{\alpha V^2 dx}{2AG} + \int \frac{T^2 dx}{2JG}$$

Castigliano's 2nd theorem

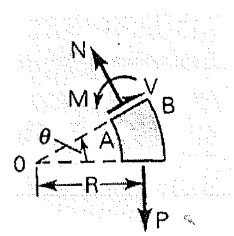
$$\partial U / \partial P_i = \delta_i$$

Castigliano's 2nd evaluation for truss structure

$$\delta_i = \frac{1}{AE} \sum_{j=1}^{n} N_j \left(\frac{\partial N_j}{\partial P_i}\right) L_j$$

Values for end-loaded curved beam

$M = PR\,(1 - \cos\theta), \quad V = P \sin\theta,$
$N = P\cos\theta$




Conditions for P.E. minimum for Rayleigh-Ritz

$$\frac{\partial \Pi}{\partial a_1} = 0, \quad \dots, \frac{\partial \Pi}{\partial a_n} = 0$$

Where $\Pi = U - W$

P.E. expression for beam

$$\Pi = \int_0^L \left[\frac{EI}{2}\left(\frac{d^2 v}{dx^2}\right)^2 - pv\right] dx$$

# MAE 416 Spring 2009 Test 1 Equation Sheet

Cylindrical Shells

$$w(x) = \frac{M_0}{2\varepsilon^2 D} e^{-\varepsilon x}(\sin \varepsilon x - \cos \varepsilon x) - \frac{Q_0}{2\varepsilon^3 D} e^{-\varepsilon x} \cos \varepsilon x + \frac{M_L}{2\varepsilon^2 D} e^{-\varepsilon (L-x)} [\sin \varepsilon (L-x) - \cos \varepsilon (L-x)] + \frac{Q_L}{2\varepsilon^3 D} e^{-\varepsilon (L-x)} \cos \varepsilon (L-x) + \frac{1}{4\varepsilon^4 D} \left[p(x) - \frac{\nu N_x}{R}\right]$$

$$\varepsilon = \left[3 (1-\nu^2)^{1/4}\right] / \sqrt{Rh}$$

$$w'(x) = \frac{dw}{dx} = \frac{M_0}{\varepsilon D} e^{-\varepsilon x} \cos \varepsilon x + \frac{Q_0}{2\varepsilon^2 D} e^{-\varepsilon x} (\sin \varepsilon x + \cos \varepsilon x) - \frac{M_L}{\varepsilon D} e^{-\varepsilon (L-x)} \cos \varepsilon (L-x) + \frac{Q_L}{2\varepsilon^2 D} e^{-\varepsilon (L-x)} [\sin \varepsilon (L-x) + \cos \varepsilon (L-x)] + \frac{1}{4\varepsilon^4 D} p'(x)$$

Singularity Functions

$$\int_{-\infty}^{x} \langle \xi - a \rangle^n \, d\xi = \frac{\langle x - a \rangle^{n+1}}{n+1} \qquad n \geq 0; (different\, rule\, for\, n < 0$$

$$S = -\int q \, dz, \qquad M = \int S \, dz, \qquad where\, \text{'q'}\, is\, applied\, loading$$

Beams

$$\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) y$$

$$\begin{Bmatrix} u'' \\ v'' \end{Bmatrix} = \frac{-1}{E(I_{xx} I_{yy} - I_{xy}^2)} \begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix} \quad or\, just\, define\, D = (I_{xx} I_{yy} - I_{xy}^2)$$

Shear Flow, Open Section

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx \, \mathrm{ds} - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty \, \mathrm{ds} \quad (assuming\, start\, \text{'s'}\, from\, edge)$$

# MAE 416 Spring 2009 Test 2 Equation Sheet

**Shear Flow Distribution--Skin only, no stringers**

Open Section

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s tx\,\mathrm{ds} - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s ty\,\mathrm{ds} \quad (assuming\ start\ 's' from\ edge)$$

Closed Section – same as above, except add $q_{s,o}$

**Skin and stringers, open section**

$$q_s = \left(\frac{S_{x,0} I_{xx} - S_{y,0} I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D x\mathrm{ds} + \sum_{r=1}^{n} B_r x_r\right) - \left(\frac{S_{y,0} I_{yy} - S_{x,0} I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D y\mathrm{ds} + \sum_{r=1}^{n} B_r y_r\right)$$

**Matching internal and external (applied) moments**

Internal moment (from shear flow) = $\oint \rho q_b ds + 2Aq_{s,o}$ where ρ is perpendicular distance and $2\mathrm{d}A = \rho\mathrm{ds}$

**Torsional stiffness, torsional moment**

Open Section

$T = GJ\,\frac{\mathrm{d}\theta}{\mathrm{dz}},\ J = \frac{1}{3}\sum_i s_i t_i^3$

Closed Section

$$\frac{\mathrm{d}\theta}{\mathrm{dz}} = \frac{T}{4A^2}\oint \frac{\mathrm{d}s}{Gt}$$

**Shear flow from torsion—closed section** $T = 2Aq$

Contribution of skin between stringers "$i$" and "$j$" to area of stringer "$i$"

$B_i = \frac{t_D b}{6}\left(2 + \frac{\sigma_j}{\sigma_i}\right)$

**Shear-deformable (Timoshenko) beam**

$S = G\,A(v' + \psi)$

$M = EI\psi'$ $\qquad$ $\psi$ *and* $v$ *are functions of* $(z)$*only*

**Admissible b.c.**

Prescribe $v$ or $S = GA\,(v' + \psi)$ AND Prescribe $\psi$ or $M = EI\psi'$

**Some leftovers—singularity functions**

$$\int_{-\infty}^{x} \langle \xi - a\rangle^n\, d\xi = \frac{\langle x - a\rangle^{n+1}}{n+1} \qquad n \geq 0; (different\ rule\ for\ n < 0$$

# MAE 416 Spring 2009 Test 3 Equation Sheet

**Skin and stringers, closed section**

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D x \mathrm{d}s + \sum_{r=1}^{n} B_r x_r\right) - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D y \mathrm{d}s + \sum_{r=1}^{n} B_r y_r\right) + q_{s,0}$$

**Matching internal and external (applied) moments**

Internal moment (from shear flow) = $\oint \rho q_b ds + 2Aq_{s,o}$ where ρ is perpendicular distance and $2\mathrm{d}A = \rho \mathrm{d}s$; (especially handy when $q_b$ is constant over a section of skin.)

**Torsional stiffness, torsional moment, closed section—single cell, then multiple cell:**

$$single \ \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{T}{4A^2}\oint \frac{\mathrm{d}s}{Gt} \ ; \ multiple \ \left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)_R = \frac{1}{2A_R}\oint_R q \frac{\mathrm{d}s}{Gt}$$

**Shear flow from torsion—closed section, single cell** $T = 2Aq$

**Contribution of skin between stringers "i" and "j" to area of stringer "i"**

$$B_i = \frac{t_D b}{6}\left(2 + \frac{\sigma_j}{\sigma_i}\right)$$

**Shear distributions, inclined booms, where "w" subscript indicates "web" (skin)**

$$S_{x,w} = S_x - \sum_{r=1}^{m} P_{z,r}\frac{dx_r}{dz} \qquad S_{y,w} = S_y - \sum_{r=1}^{m} P_{z,r}\frac{dy_r}{dz}$$

Balance of internal and external moments, where "ρ" is lever arm:

$$S_x\eta_0 - S_y\xi_0 = \oint q_\mathrm{b}\rho \mathrm{d}s + 2Aq_{s,0} - \sum_{r=1}^{m} P_{x,r}\eta_r + \sum_{r=1}^{m} P_{y,r}\xi_r$$

**Sum of torques, multi-cell:**

$$T = \sum_{r}^{N} 2A_r q_r$$

**Potential energy, buckling & bending, at neutral equilibrium**

$$buckling \ \Pi = \frac{EI}{2}\int_0^L (v'')^2 dz - \frac{P_{cr}}{2}\int_0^L (v')^2 \ dz \ ; bending \ \Pi = \frac{EI}{2}\int_0^L (v'')^2 dz - \int_0^L q \ v \ dz$$

**General solution for buckling equation, and its derivatives & force and moment relationships**

$$v(z) = C_1 + C_2 z + C_3 \sin\lambda z + C_4 \cos\lambda z, \qquad \lambda = \sqrt{\frac{P}{EI}}$$

$$v'(z) = C_2 + C_3\lambda\cos\lambda z - C_4\lambda\sin\lambda z$$

$$v''(z) = -C_3\lambda^2\sin\lambda z - C_4\lambda^2\cos\lambda z$$
$$v'''(z) = -C_3\lambda^3\cos\lambda z + C_4\lambda^3\sin\lambda z$$

$$Moment \ \& \ Shear \ relationships\!: \ M = -EIv'' \quad S = -EIv''' - Pv'$$